THE DEVIATIONS OF FALLING BODIES.

By F. R. Moulton.*

1. Introduction. The subject of the deviations of freely falling bodies has been considered in many memoirs from the time of Laplace and Gauss to the recent work of Roever and Woodward.† All writers have agreed that a body falling from rest near the surface of the earth will deviate to the eastward with respect to a plumb-line hung from the initial point, but a great variety of results have been obtained regarding the deviation measured along the meridian. For example, Laplace found no meridional deviation, Gauss a small deviation toward the equator, Roever a deviation toward the equator several times that of Gauss, and Woodward a small deviation away from the equator. The diversity of these results is the occasion for this paper.

In the present paper the solution of the problem of the deviations of falling bodies is made to depend upon two well-established mathematical theories, viz., that of the solution of analytic differential equations in the neighborhood of regular points, and that of implicit analytic functions. The methods employed are rigorous, of general applicability, and relatively simple. The problem involves three parameters, viz., the height from which the body falls, the rate of rotation of the rotating body, and the oblateness of the rotating body. Only the first of these is arbitrary, and it is shown that the deviations of the falling body can be developed as converging power series in it if it is sufficiently small. In the meridional deviation the coefficient of the first power of this parameter, which is a function of the other two parameters and can be developed as a power series in them, is identically zero, at least in the case where the rotating body is a figure of revolution; but the coefficient of the second power of the parameter is not zero. The sign of this coefficient rigorously determines the character of the deviation when the distance through which the body falls is not too great. The principal part of the coefficient of the second power of the parameter agrees exactly with the result found by Roever in his first paper by an altogether different method. Differences in terms of higher order

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† For a more complete history of the question consult Roever's papers: Trans. of the Am. Math. Soc., vol. XII (1911), pp. 335–353, and vol. XIII (1912), pp. 469–490; and Woodward's articles: The Astron. Jour., vol. XXVIII (1913), pp. 17–29, and Science, vol. XXXVIII (1913), pp. 315–319.

would be found if they were computed because his definitions of the deviations differ somewhat from those used in this paper.

The question naturally arises why so great a variety of results has been obtained. The answer seems to lie in the fact that the solution of the problem depends upon two distinct mathematical processes, however much they may be concealed by the details of treatment; that both of them naturally lead to power series in all three parameters; that the coefficients of all terms of the first degree in the first parameter are identically zero; and that methods of approximation are not adapted to establishing identities. It is clear from what has been said that the correct result could never be established by explicitly developing the results in all three parameters because only a finite number of terms could be computed; and it is equally clear that great care would be necessary to secure exactly the same approximation in solving the differential equations and the implicit functions by approximating methods because the two processes are so different. Most writers have developed the results from the beginning in all three parameters, and have used approximations without proving their legitimacy.

2. The Differential Equations. Take the origin at the center of the rotating body and the principal polar axis coincident with the axis of rotation. Then, if r represents the distance, φ the latitude, $\theta = \theta_0 + \omega t + \lambda$ (where ω is the rate of rotation of the body) the longitude, and V the potential function, the differential equations of motion are

$$
(1)\qquad
\begin{aligned}
&\frac{d^2 r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 - r\left(\omega + \frac{d\lambda}{dt}\right)^2 \cos^2\varphi = \frac{\partial V}{\partial r},\\
&\frac{d}{dt}\left(r^2 \frac{d\varphi}{dt}\right) + r^2\left(\omega + \frac{d\lambda}{dt}\right)^2 \sin\varphi\cos\varphi = \frac{dV}{d\varphi},\\
&\frac{d}{dt}\left[r^2\left(\omega + \frac{d\lambda}{dt}\right)\cos^2\varphi\right] = \frac{\partial V}{\partial \lambda}.
\end{aligned}
$$

The potential function V in all cases can be developed as an infinite series of spherical harmonics whose coefficients depend upon the distribution of mass of the rotating body. If the rotating body is a figure of revolution about the axis of rotation whose density does not depend upon the longitude, the function V can be developed as a series of zonal harmonics of the form

$$
(2)\qquad V = \frac{\alpha}{r} + \frac{\beta}{r^3}(1 - 3\sin^2\varphi) + \cdots.
$$

Only the first two terms have sensible effects upon falling bodies and plumb-lines for short distances.

Let P_0 be the point from which the body falls and from which the plumb-line is suspended, $P_0^{(0)}$ the point where the line from the origin

to P_0 pierces the surface of the rotating body, P_1 the point where the falling body strikes the surface of the rotating body, and P_2 the point where the plumb-bob touches the surface of the rotating body. For notation

$$
(3)\qquad
\begin{aligned}
&\text{let the coördinates of } P_0 \text{ be } r_0(1+h),\ \ \varphi_0,\ \ 0,\\
&\text{let the coördinates of } P_0^{(0)} \text{ be } r_0,\ \ \varphi_0,\ \ 0,\\
&\text{let the coördinates of } P_1 \text{ be } r_1,\ \ \varphi_1,\ \ \lambda_1,\\
&\text{let the coördinates of } P_2 \text{ be } r_2,\ \ \varphi_2,\ \ \lambda_2.
\end{aligned}
$$

The differential equations (1) are to be solved subject to the initial conditions

$$
(4)\qquad
\begin{aligned}
r(0) &= r_0(1+h), & \varphi(0) &= \varphi_0, & \lambda(0) &= 0,\\
r'(0) &= 0, & \varphi'(0) &= 0, & \lambda'(0) &= 0.
\end{aligned}
$$

The problem is to find the points P_1 and P_2. If V is a sum of zonal harmonics, $\lambda_2 = 0$. When P_1 and P_2 have been found the longitudinal and meridional deviations in angular measure are respectively $\lambda_1 - \lambda_2$ and $\varphi_1 - \varphi_2$. If the first is positive the deviation is to the east, and if the second is positive the deviation is from the equator.

3. Integration of Equations (1). Since equations (1) are regular in the vicinity of the initial conditions, it follows from the general theory of analytic differential equations that they can be integrated uniquely as power series in t which converge for $|t|$ sufficiently small. It follows from (4) that the solution has no terms of the first degree in t. Hence the solution in the general case in which V is a sum of spherical harmonics has the form

$$
(5)\qquad
\begin{aligned}
r &= r_0(1+h) + 0t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots,\\
\varphi &= \varphi_0 + 0t + b_2t^2 + b_3t^3 + b_4t^4 + \cdots,\\
\lambda &= 0 + 0t + c_2t^2 + c_3t^3 + c_4t^4 + \cdots,
\end{aligned}
$$

where the a_i, b_i, and c_i are constants which can be obtained by substituting (5) in (1) and equating coefficients of corresponding powers of t.

But if V is a sum of zonal harmonics, and consequently independent of λ, it follows from the third equation of (1) that $c_2 = 0$; and then from the first two equations of (1) that $a_3 = b_3 = 0$. Consequently, in this case the solution has the form

$$
(6)\qquad
\begin{aligned}
r &= r_0(1+h) + a_2t^2 + 0t^3 + a_4t^4 + a_5t^5 + \cdots,\\
\varphi &= \varphi_0 + b_2t^2 + 0t^3 + b_4t^4 + b_5t^5 + \cdots,\\
\lambda &= 0 + 0 + c_3t^3 + c_4t^4 + c_5t^5 + \cdots.
\end{aligned}
$$

In general, the coefficients of all powers of t beyond the third are distinct from zero.

The full details will be given only in the case where V is a sum of zonal harmonics. It is found that

$$
\begin{aligned}
2a_2 &= \omega^2 r_0(1+h)\cos^2\varphi_0 + \left(\frac{\partial V}{\partial r}\right)_0, \\
2b_2 &= -\tfrac{1}{2}\omega^2 \sin 2\varphi_0 - \frac{1}{r_0^2(1+h)^2}\left(\frac{\partial V}{\partial \varphi}\right)_0, \\
6c_3 &= -4\omega\left[\frac{a_2}{r_0(1+h)} - b_2\tan\varphi_0\right], \\
12a_4 &= 4r_0(1+h)b_2^2 + \omega^2\cos^2\varphi_0\, a_2 - r_0(1+h)\omega^2 \sin 2\varphi_0 b_2 \\
&\quad + 6r_0(1+h)\cos^2\varphi_0 c_3 + \left(\frac{\partial^2 V}{\partial r^2}\right)_0 a_2 + \left(\frac{\partial^2 V}{\partial r\partial\varphi}\right)_0 b_2, \\
12b_4 &= -\frac{12a_2b_2}{r_0(1+h)} - \frac{\omega^2\sin 2\varphi_0}{r_0(1+h)}a_2 - 3\omega\sin 2\varphi_0 c_3 - \omega^2\cos 2\varphi_0 b_2 \\
&\quad + \left(\frac{\partial^2 V}{\partial r\partial\varphi}\right)_0 a_2 + \left(\frac{\partial^2 V}{\partial \varphi^2}\right)_0 b_2.
\end{aligned} \tag{7}
$$

The coefficients of higher powers of t are not needed.

4. Determination of the Coördinates of P_1. Let t_1 be the time at which the falling body reaches the surface of the rotating body. Then the coördinates of P_1 are obtained by replacing t by t_1 in equations (6). Hence the value of t_1 must be found.

The point P_1 is subject to the condition that it shall be on the surface of the rotating body, which is an equipotential surface for the potential function V and the rotational potential $\frac{1}{2}\omega^2 r^2\cos^2\varphi$. Hence the coördinates of P_1 satisfy the equation

$$
\tfrac{1}{2}\omega^2 r_1^2\cos^2\varphi_1 + \frac{\alpha}{r_1} + \frac{\beta}{r_1^3}(1 - 3\sin^2\varphi_1) + \cdots = C,
$$

where C is determined by any point on the surface. But the coördinates of $P_0^{(0)}$ also lie on the surface of the rotating body and satisfy an equation of the same form. The difference of these equations, which eliminates the constant C, is

$$
\begin{aligned}
F &\equiv \tfrac{1}{2}\omega^2(r_1^2\cos^2\varphi_1 - r_0^2\cos^2\varphi_0) + \alpha\left(\frac{1}{r_1} - \frac{1}{r_0}\right) + \beta\left(\frac{1}{r_1^3} - \frac{1}{r_0^3}\right) \\
&\quad - 3\beta\left(\frac{\sin^2\varphi_1}{r_1^3} - \frac{\sin^2\varphi_0}{r_0^3}\right) + \cdots = 0.
\end{aligned} \tag{8}
$$

If t_1 is put in (6) in place of t, and if the series for r_1 and φ_1 are substituted in (8), the resulting expression can be developed as a power series

in t_1 and h which will converge provided $|t_1|$ is sufficiently small and $|h| < 1$. That is, (8) takes the form

$$F = p(h, t_1) = 0, \tag{9}$$

where p is a power series in h and t_1. Moreover, it is easily found that

$$p(0, 0) = 0, \quad \left(\frac{\partial p}{\partial t_1}\right)_0 = 0, \quad \left(\frac{\partial^2 p}{\partial t_1^2}\right)_0 = 4[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2], \tag{10}$$

where $a_2^{(0)}$ and $b_2^{(0)}$ are obtained from a_2 and b_2 by putting h equal to zero. It follows from the theory of implicit functions that (9) is solvable for t_1 as a power series in $\sqrt{h}$, vanishing for $h = 0$ and converging for $|h|$ sufficiently small, provided $4[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2]$ is distinct from zero. But this quantity is the square of the acceleration of gravity at the point $P_0^{(0)}$, and consequently is not zero. Therefore the solution of (9) has the form

$$t_1 = \sqrt{h}\, p_1(\sqrt{h}), \tag{11}$$

where $p_1(\sqrt{h})$ is a power series in $\sqrt{h}$.

It has been stated that it is sufficient to have the developments of series (6) to terms of the fourth degree inclusive in t. If the series are terminated at this point and V is a sum of zonal harmonics, F becomes a function of t_1^2 and h, and equation (8) can be solved for t_1^2 as a power series in h of the form

$$t_1^2 = \alpha_1 h + \alpha_2 h^2 + \cdots. \tag{12}$$

The first two terms of this series, which alone are required, are identical with the first two terms of the square of (11).

In order to find the coefficients α_1 and α_2 the explicit form of (9) is required. It is found from (6), (7), and (8) that

$$\begin{aligned}
F &= a_{10}t_1^2 + a_{01}h + a_{20}t_1^4 + a_{11}t_1^2 h + a_{02}h^2 + \cdots = 0,\\
a_{10} &= 2[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2],\\
a_{01} &= 2r_0 a_2^{(0)},\\
a_{20} &= 2a_2^{(0)}a_4^{(0)} + 2r_0^2 b_2^{(0)} b_4^{(0)}\\
&\quad + \left[\tfrac{1}{2}\omega^2\cos^2\varphi_0 + \frac{\alpha}{r_0^3} + \frac{6\beta}{r_0^5}(1 - 3\sin^2\varphi_0)\right](a_2^{(0)})^2\\
&\quad - \left[r_0\omega^2 - \frac{9\beta}{r_0^4}\right]\sin 2\varphi_0 a_2^{(0)}b_2^{(0)} - \left[\tfrac{1}{2}r_0^2\omega^2 + \frac{3\beta}{r_0^3}\right]a_2^{(0)}b_2^{(0)},\\
a_{11} &= \left[r_0\omega^2\cos^2\varphi_0 + \frac{2\alpha}{r_0^2} + \frac{12\beta}{r_0^4}(1 - 3\sin^2\varphi_0) + 2\left(\frac{\partial a_2}{\partial h}\right)_0\right]a_2^{(0)}\\
&\quad + \left[-r_0^2\omega^2\sin 2\varphi_0 + \frac{9\beta}{r_0^3}\sin 2\varphi_0 + 2r_0^2\left(\frac{\partial b_2}{\partial h}\right)_0\right]b_2^{(0)},\\
a_{02} &= \tfrac{1}{2}r_0^2\omega^2\cos^2\varphi_0 + \frac{\alpha}{r_0} + \frac{6\beta}{r_0^3}(1 - 3\sin^2\varphi_0).
\end{aligned} \tag{13}$$

On substituting (12) in the first of (13) and equating coefficients of corresponding powers of h, it is found that

$$(14)\quad \alpha_1 = \frac{-r_0 a_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2}, \qquad \alpha_2 = -\frac{[a_{10}^2 a_{02} - a_{11}a_{01}a_{10} + a_{01}^2 a_{20}]}{a_{10}^3}.$$

On substituting (12) in the second and third equations of (6), it is found that

$$(15)\quad \begin{aligned} \varphi_1 &= \varphi_0 - \frac{r_0 a_2^{(0)} b_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2} h + \left[\alpha_1 \left(\frac{\partial b_2}{\partial h}\right)_0 + \alpha_2 b_2^{(0)} \right. \\ &\qquad\qquad \left. + \alpha_1^2 b_4^{(0)}\right] h^2 + \cdots, \\ \lambda_1 &= 0 + \left\{\frac{-r_0 a_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2}\right\}^{\frac{3}{2}} c_3^{(0)} h^{\frac{3}{2}} + \cdots. \end{aligned}$$

5. Determination of the Coördinates of P_2. The point P_2 is determined by the conditions that it shall lie on the surface of the rotating body and that the normal to the equipotential surface passing through it shall also pass through P_0. The first of these conditions gives an equation which differs from (8) only in that the subscript 2 appears in place of 1.

Let

$$(16)\quad W = \tfrac{1}{2}\omega^2 r^2 \cos^2\varphi + V.$$

Then the equations of the normal to the equipotential surface $W = C$ at the point $P_2(x_2, y_2, z_2)$ are

$$(17)\quad \frac{\left(\frac{\partial W}{\partial x}\right)_2}{x - x_2} = \frac{\left(\frac{\partial W}{\partial y}\right)_2}{y - y_2} = \frac{\left(\frac{\partial W}{\partial z}\right)_2}{z - z_2}.$$

If this line passes through P_0 these equations are satisfied by

$$(18)\quad \begin{aligned} x &= x_0 = r_0(1 + h)\cos\varphi_0 \cos\theta_0, \\ y &= y_0 = r_0(1 + h)\cos\varphi_0 \sin\theta_0, \\ z &= z_0 = r_0(1 + h)\sin\varphi_0. \end{aligned}$$

The two equations (17) and the one which expresses the condition that P_2 shall be on the surface of the rotating body are sufficient to determine x_2, y_2, and z_2, or preferably their equivalents in polar coördinates.

Suppose V is a sum of zonal harmonics. Then the two equations (17) can be replaced by a single one, which is, if θ_0 is taken equal to zero,

$$(z_0 - z_2)\left(\frac{\partial W}{\partial x}\right)_2 - (x_0 - x_2)\left(\frac{\partial W}{\partial z}\right)_2 = 0.$$

The equation which expresses the condition that P_2 shall be on the surface $W = C$ and this equation become respectively in polar coördinates, after some simplifications,

$$
\begin{aligned}
F_2 \equiv\ & \tfrac{1}{2}\omega^2(r_2^2 \cos^2 \varphi_2 - r_0^2 \cos^2 \varphi_0) + \alpha\left(\frac{1}{r_2} - \frac{1}{r_0}\right) \\
& + \beta\left(\frac{1}{r_2^3} - \frac{1}{r_0^3}\right) - 3\beta\left(\frac{\sin^2 \varphi_2}{r_2^3} - \frac{\sin^2 \varphi_0}{r_0^3}\right) = 0,
\end{aligned}
$$

$$
\begin{aligned}
(19)\qquad G_2 \equiv\ & \frac{r_0(1+h)}{r_2^2}\left[\alpha + \frac{3\beta}{r_2^2}(1 - 5\sin^2 \varphi_2)\right] \sin(\varphi_2 - \varphi_0) \\
& + \frac{6\beta}{r_2^4}[r_0(1+h)\cos \varphi_0 - r_2 \cos \varphi_2] \sin \varphi_2 \\
& + r_2\omega^2[r_0(1+h)\sin \varphi_0 - r_2 \sin \varphi_2] \cos \varphi_2 = 0,
\end{aligned}
$$

from which r_2 and φ_2 can be determined.

In order to solve (19) it is convenient to let

$$(20)\qquad r_2 = r_0(1 + \rho), \quad \varphi_2 = \varphi_0 + \sigma.$$

Then it follows that F_2 and G_2 can be expanded as power series in ρ, σ, and h of the form

$$(21)\qquad F_2 = f_2(\rho, \sigma, h) = 0, \quad G_2 = g_2(\rho, \sigma, h) = 0.$$

It is noted that f_2 does not involve h, g_2 contains h linearly, and both f_2 and g_2 converge for any finite values of σ and h provided $|\rho| < 1$.

Both of equations (21) are satisfied by $\rho = \sigma = h = 0$. The condition that they shall be uniquely solvable for ρ and σ as power series in h, vanishing for $h = 0$, is that the Jacobian of f_2 and g_2 with respect to ρ and σ shall be distinct from zero for $\rho = \sigma = h = 0$. It is found from (19) that

$$(22)\qquad J = \begin{vmatrix} \dfrac{\partial f_2}{\partial \rho}, & \dfrac{\partial f_2}{\partial \sigma} \\[2ex] \dfrac{\partial g_2}{\partial \rho}, & \dfrac{\partial g_2}{\partial \sigma} \end{vmatrix} = \begin{vmatrix} 2r_0 a_2^{(0)}, & +2r_0^2 b_2^{(0)} \\ 2r_0^2 b_2^{(0)}, & -2r_0 a_2^{(0)} \end{vmatrix} = -4r_0^2[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2],$$

which is distinct from zero because it is $-r_0^2$ times the square of the acceleration of gravity at $P_0^{(0)}$. Therefore the solution of (21) has the form

$$(23)\qquad \rho = \beta_1 h + \beta_2 h^2 + \cdots, \quad \sigma = \gamma_1 h + \gamma_2 h^2 + \cdots,$$

which converge provided $|h|$ is sufficiently small.

In order to determine the coefficients of (23) it is necessary to have the explicit expansions of (21). On referring to (19) it is seen that these series have the form

$$
(24)\quad \begin{aligned}
F_2 &= b_{100}\rho + b_{010}\sigma + b_{200}\rho^2 + b_{110}\rho\sigma + b_{020}\sigma^2 + \cdots = 0,\\
G_2 &= c_{100}\rho + c_{010}\sigma + c_{001}h + c_{200}\rho^2 + c_{110}\rho\sigma\\
&\qquad + c_{101}\rho h + c_{020}\sigma^2 + c_{011}\sigma h + \cdots = 0,
\end{aligned}
$$

where all terms which are not zero up to the second order in ρ, σ, and h have been written. It is easily found from equations (19) that the explicit values of the coefficients of (24) are

$$
(25)\quad \begin{aligned}
b_{100} &= 2r_0 a_2^{(0)}, \qquad b_{010} = 2r_0^2 b_2^{(0)},\\
b_{200} &= \tfrac{1}{2}r_0^2\omega^2\cos^2\varphi_0 + \frac{\alpha}{r_0} + \frac{6\beta}{r_0^3}(1 - 3\sin^2\varphi_0),\\
b_{110} &= \left[-r_0^2\omega^2 + \frac{9\beta}{r_0^3}\right]\sin 2\varphi_0,\\
b_{020} &= -\left[\tfrac{1}{2}r_0^2\omega^2 + \frac{3\beta}{r_0^3}\right]\cos 2\varphi_0,\\
c_{100} &= 2r_0^2 b_2^{(0)}, \qquad c_{010} = -2r_0 a_2^{(0)},\\
c_{001} &= -2r_0^2 b_2^{(0)}, \qquad c_{200} = \left[-\tfrac{1}{2}r_0^2\omega^2 + \frac{12\beta}{r_0^3}\right]\sin 2\varphi_0,\\
c_{110} &= -r_0^2\omega^2(2 - 3\sin^2\varphi_0) - \frac{2\alpha}{r_0} - \frac{6\beta}{r_0^3}(3 - 8\sin^2\varphi_0),\\
c_{101} &= \left[\tfrac{1}{2}r_0^2\omega^2 - \frac{12\beta}{r_0^3}\right]\sin 2\varphi_0,\\
c_{020} &= \left[\tfrac{3}{4}r_0^2\omega^2 - \frac{21}{2}\frac{\beta}{r_0^3}\right]\sin 2\varphi_0,\\
c_{011} &= -r_0^2\omega^2\sin^2\varphi_0 + \frac{\alpha}{r_0} + \frac{3\beta}{r_0^3}(3 - 7\sin^2\varphi_0).
\end{aligned}
$$

In order to obtain the meridional deviation it is sufficient to compute γ_1 and γ_2, but in order to obtain γ_2 it is necessary to compute β_1. It is found on substituting equations (23) in (24) and equating to zero the coefficients of corresponding powers of h, and then solving the resulting linear equations, that

$$
(26)\quad \begin{aligned}
\beta_1 &= \frac{r_0^2(b_2^{(0)})^2}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2},\\
\gamma_1 &= \frac{-r_0 a_2^{(0)} b_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2},\\
D\gamma_2 &= -b_{100}c_{011}\gamma_1 - (b_{100}c_{200} - c_{100}b_{200})\beta_1^2\\
&\quad - (b_{100}c_{110} - c_{100}b_{110})\beta_1\gamma_1 - (b_{100}c_{020} - c_{100}b_{020})\gamma_1^2 - b_{100}c_{101}\beta_1,
\end{aligned}
$$

where

(27) $$D = -4r_0^2[(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2].$$

These results substituted in (23) and (20) give the coördinates of P_2 when V is a sum of zonal harmonics.

6. Expressions for the Deviations. In case V is a sum of zonal harmonics λ_2 is zero, and it follows from the second equation of (15) that the expression for the angular deviation in longitude is

(28) $$\lambda_1 = \left\{\frac{-r_0 a_2^{(0)}}{(a_2^{(0)})^2 + r_0^2(b_2^{(0)})^2}\right\}^{\frac{3}{2}} c_3^{(0)} h^{\frac{3}{2}} + \cdots.$$

The value of h, which is entirely arbitrary, can be taken so small that the sign of λ will be determined by the first term of this series.

It follows from (7) and (2) that

(29 $$\begin{aligned} a_2^{(0)} &= -\frac{\alpha}{2r_0^2} + \tfrac{1}{2}r_0\,\omega^2\cos^2\varphi_0 - \frac{3\beta}{2r_0^4}(1 - 3\sin^2\varphi_0), \\ b_2^{(0)} &= -\tfrac{1}{4}\omega^2\sin 2\varphi_0 - \frac{3\beta}{2r_0^5}\sin 2\varphi_0. \end{aligned}$$

The quantity $a_2^{(0)}$ is always negative for an actual physical body for otherwise there would be an acceleration outward along the radius at $P_0^{(0)}$. In the case of the earth the first term of the expression for $a_2^{(0)}$ is numerically about 300 times the coefficient of $\cos^2\varphi_0$ in the second term, and about 600 times the coefficient of $(1 - 3\sin^2\varphi_0)$ in the third term. That is,

(30) $$\frac{\alpha}{r_0^2} = 300 r_0\omega^2 = 600\frac{3\beta}{r_0^4}$$

approximately. Consequently

(31) $$\begin{aligned} c_3^{(0)} &= -\tfrac{2}{3}\omega\left[\frac{a_2^{(0)}}{r_0} - b_2^{(0)}\tan\varphi_0\right] \\ &= \tfrac{2}{3}\omega\left[\frac{\alpha}{2r_0^3} - \tfrac{1}{2}\omega^2 + \frac{3\beta}{2r_0^5}(1 - 5\sin^2\varphi_0)\right] \end{aligned}$$

is positive. Therefore the deviation is toward the eastward. Since $b_2^{(0)}$ is small compared to $a_2^{(0)}$, an approximate expression for the easterly deviation, obtained by neglecting $b_2^{(0)}$ in the denominator of (28) and the terms in $a_2^{(0)}$ and $c_3^{(0)}$ which are multiplied by ω^2 or β, is

(32) $$\lambda_1 = \left(\frac{2\sqrt{2}}{3}\frac{r_0^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}}\omega + \cdots\right)h^{\frac{3}{2}} + \cdots.$$

By making use of (23) and the explicit value of γ_1 given in (26), the meridional deviation is found from (15) and (20) to be

$$(33)\quad \varphi_1 - \varphi_2 = 0h + \left[\alpha_1 \left(\frac{\partial b_2}{\partial h} \right)_0 + \alpha_2 b_2^{(0)} + \alpha_1^2 b_4^{(0)} - \gamma_2 \right] h^2 + \cdots.$$

The first important fact to be noted is that the coefficient of h is identically zero. It follows from the first equation of (15) and from (29) and (30) that the numerator and denominator of the coefficient of h in the expression for φ_1 are polynomials in ω^2 and β, and that the quotient can be expanded as an infinite converging power series in ω^2 and β. Exactly the same is of course true for the coefficient of h in the expression for φ_2. The quantities φ_1 and φ_2 are derived by quite different processes, the former involving the solution of differential equations and the latter only the solution of implicit functions. Hence it is clear that methods of approximation in carrying out these different processes are beset with danger because the effects upon the results are apt not to be the same in both.

The coefficient of h^2 in (33) is exact, but in order to determine its sign all terms except those of lowest order in ω^2 and β may be neglected. It follows from the numerical coefficients which are involved and the relations given in (30) that, at least for the earth, these simplifications can not change the sign of the result. It follows from (29) that $a_2^{(0)}$ is of order zero in ω^2 and β, and that $b_2^{(0)}$ is of order one in these same quantities. Then it is seen from (7) that $b_4^{(0)}$ and $(\partial b_2/\partial h)_0$ are of order one in ω^2 and β, though most of the terms in $b_4^{(0)}$ are of higher order. They are explicitly

$$(34)\quad \begin{aligned} b_4^{(0)} &= -\frac{\alpha}{24 r_0^3}\left(4\omega^2 + \frac{27\beta}{r_0^5}\right)\sin 2\varphi_0 + \text{terms of the second order}, \\ \left(\frac{\partial b_2}{\partial h}\right)_0 &= +\frac{15\beta}{2 r_0^5}\sin 2\varphi_0. \end{aligned}$$

It follows from (14) that α_1 and α_2 are of order zero in ω^2 and β, and from the first two equations of (26) that β_1 is of the second order and γ_1 is of the first order.

Now consider γ_2. Since D is of order zero all the terms in the right member of the third equation of (26), except possibly the first, are of the second order at least. It is seen from the first and last equations of (25) that b_{100} and c_{011} are of order zero. Therefore the first term of γ_2 is of order one.

On retaining only the terms of the first order in the coefficient of h^2 in (33), it is found that the approximate expression for the meridional deviation is

$$\varphi_1 - \varphi_2 = \left[\frac{1}{2} \frac{r_0^3}{\alpha} \left(\frac{5\beta}{r_0^5} - 4\omega^2 \right) \sin 2\varphi_0 + \cdots \right] h^2 + \cdots. \tag{35}$$

For the earth $4\omega^2$ is more than five times as great as $5\beta/r_0^5$. Therefore *the deviation of a body freely falling a small distance near the earth's surface is equatorward for all latitudes between* 0 *and* $\pm$ 90°.

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